

# LOEWNER'S TORUS INEQUALITY WITH ISOSYSTOLIC DEFECT

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ABSTRACT. We show that Bonnesen's isoperimetric defect has a systolic analog for Loewner's torus inequality. The isosystolic defect is expressed in terms of the probabilistic variance of the conformal factor of the metric  $\mathcal{G}$  with respect to the flat metric of unit area in the conformal class of  $\mathcal{G}$ .

## CONTENTS

1. Bonnesen defect and isosystolic defect	1
2. Variance, Hermite constant, successive minima	5
3. Standard fundamental domain and Eisenstein integers	6
4. Fundamental domain and Loewner's torus inequality	7
5. First fundamental form and surfaces of revolution	9
6. A second isosystolic defect term	10
7. Biaxial projection and second defect	12
8. Acknowledgments	13
References	14

## 1. BONNESEN DEFECT AND ISOSYSTOLIC DEFECT

The *systole* of a compact metric space  $X$  is a metric invariant of  $X$ , defined to be the least length of a noncontractible loop in  $X$ . We will denote it  $\text{sys} = \text{sys}(X)$ , *cf.* M. Gromov [Gr83, Gr96, Gr99, Gr07]. When  $X$  is a graph, the invariant is usually referred to as the *girth*, ever since W. Tutte's article [Tu47]. Possibly inspired by the latter, C. Loewner started thinking about systolic questions on surfaces in the

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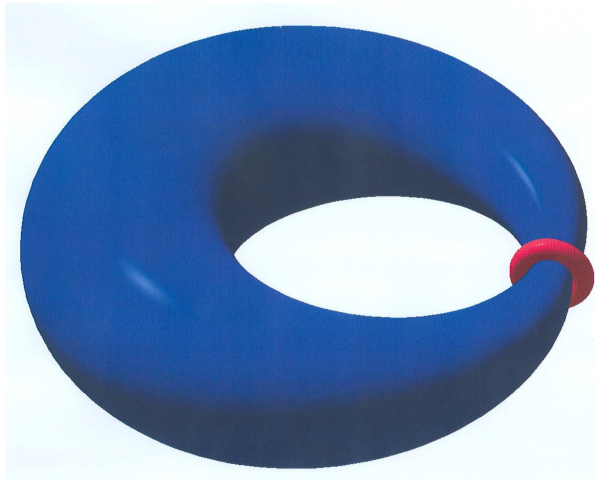


FIGURE 1.1. Wikiartist's conception of a shortest loop on a torus

late forties, resulting in a '50 thesis by his student P.M. Pu, published as [Pu52].

Loewner himself did not publish his torus inequality (1.1), apparently leaving it to Pu to pursue this line of research. Meanwhile, the latter was recalled to the mainland after the communists ousted Chiang Kai-shek in '49. Pu was henceforth confined to research in fuzzy topology in the service of the people. Our guess is that Pu may have otherwise obtained a geometric inequality with isosystolic defect, already half a century ago, placing it among the classics of the global geometry of surfaces.

Similarly to the isoperimetric inequality, Loewner's torus inequality relates the total area, to a suitable 1-dimensional invariant, namely the systole, *i.e.* least length of a noncontractible loop on the torus  $(\mathbb{T}^2, \mathcal{G})$ :

$$\text{area}(\mathcal{G}) - \frac{\sqrt{3}}{2} \text{sys}(\mathcal{G})^2 \geq 0, \quad (1.1)$$

*cf.* (4.3) and [Pu52, Ka07].

The classical Bonnesen inequality [Bo21] is the strengthened isoperimetric inequality

$$L^2 - 4\pi A \geq \pi^2(R - r)^2, \quad (1.2)$$

see [BZ88, p. 3]. Here  $A$  is the area of the region bounded by a closed Jordan curve of length (perimeter)  $L$  in the plane,  $R$  is the circumradius of the bounded region, and  $r$  is its inradius. The error term  $\pi^2(R - r)^2$  on the right hand side of (1.2) is traditionally referred to as the *isoperimetric defect*.

In the present text, we will strengthen Loewner's torus inequality by introducing a "defect" term à la Bonnesen. There is no defect term in either [Pu52] or [Ka07]. The approach that has been used in the literature is via an integral identity expressing area in terms of energies of loops. Somehow researchers in the field seem to have overlooked the fact that the computational formula for the variance yields an improvement, namely the defect term. There is thus a significant change of focus, from the integral geometric identity, to the application of the computational formula, elementary though it may be.

If we use conformal representation to express the metric  $\mathcal{G}$  on the torus as

$$f^2(dx^2 + dy^2)$$

with respect to a unit area flat metric  $dx^2 + dy^2$  on the torus viewed as a quotient of the  $(x, y)$  plane by a lattice (see (1.6)), then the defect term in question is simply the variance of the conformal factor  $f$  above. Then the inequality with the defect term can be written as follows:

$$\text{area}(\mathcal{G}) - \frac{\sqrt{3}}{2} \text{sys}(\mathcal{G})^2 \geq \text{Var}(f). \quad (1.3)$$

Here the error term, or *isosystolic defect*, is given by the variance

$$\text{Var}(f) = \int_{\mathbb{T}^2} (f - m)^2 \quad (1.4)$$

of the conformal factor  $f$  of the metric  $\mathcal{G} = f^2(dx^2 + dy^2)$  on the torus, relative to the unit area flat metric  $\mathcal{G}_0 = dx^2 + dy^2$  in the same conformal class. Here

$$m = \int_{\mathbb{T}^2} f \quad (1.5)$$

is the mean of  $f$ . More concretely, if  $(\mathbb{T}^2, \mathcal{G}_0) = \mathbb{R}^2/L$  where  $L$  is a lattice of unit coarea, and  $D$  is a fundamental domain for the action of  $L$  on  $\mathbb{R}^2$  by translations, then the integral (1.5) can be written as

$$m = \int_D f(x, y) dx dy$$

where  $dx dy$  is the standard measure of  $\mathbb{R}^2$ . Every flat torus is isometric to a quotient  $\mathbb{T}^2 = \mathbb{R}^2/L$  where  $L$  is a lattice, cf. [Lo71, Theorem 38.2]. Recall that the uniformisation theorem in the genus 1 case can be formulated as follows.

**Theorem 1.1** (Uniformisation theorem). *For every metric  $\mathcal{G}$  on the 2-torus  $\mathbb{T}^2$ , there exists a lattice  $L \subset \mathbb{R}^2$  and a positive  $L$ -periodic function  $f(x, y)$  on  $\mathbb{R}^2$  such that the torus  $(\mathbb{T}^2, \mathcal{G})$  is isometric to*

$$(\mathbb{R}^2/L, f^2 ds^2), \quad (1.6)$$

where  $ds^2 = dx^2 + dy^2$  is the standard flat metric of  $\mathbb{R}^2$ .

When the flat metric is that of the unit square torus, Loewner's inequality can be strengthened to the inequality

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f),$$

cf. (4.4). In this case, if the conformal factor depends only on one variable (as, for example, in the case of surfaces of revolution), one can strengthen the inequality further by providing a second defect term as follows:

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f) + \frac{1}{4} |f_0|_1^2, \quad (1.7)$$

where  $f_0 = f - E(f)$ , while  $E(f)$  is the expected value of  $f$ , and  $|\cdot|_1$  is the  $L^1$ -norm. See also inequality (6.2). More generally, we obtain the following theorem.

We first define a “biaxial” projection  $\mathbb{P}_{\text{BA}}(f)$  as follows. Given a doubly periodic function  $f(x, y)$ , *i.e.* a function defined on  $\mathbb{R}^2/\mathbb{Z}^2$ , we decompose  $f$  by setting

$$f(x, y) = E(f) + g_f(x) + h_f(y) + k_f(x, y),$$

where the single-variable functions  $g_f$  and  $h_f$  have zero means, while  $k_f$  has zero mean along every vertical and horizontal unit interval. We have  $g_f(x) = \int_0^1 f(x, y) dy$ , while  $h_f(y) = \int_0^1 f(x, y) dx$ . The projection  $\mathbb{P}_{\text{BA}}(f)$  is then defined by setting

$$\mathbb{P}_{\text{BA}}(f) = g_f(x) + h_f(y).$$

In terms of the double Fourier series of  $f$ , the projection  $\mathbb{P}_{\text{BA}}$  amounts to extracting the  $(m, n)$ -terms such that  $mn = 0$  (*i.e.* the terms located along the pair of coordinate axes), but  $(m, n) \neq (0, 0)$ .

**Theorem 1.2.** *In the conformal class of the unit square torus, the metric  $f^2 ds^2$  defined by a general conformal factor  $f(x, y) > 0$ , satisfies the following version of Loewner's torus inequality with a second systolic defect term:*

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f) + \frac{1}{16} |\mathbb{P}_{\text{BA}}(f)|_1^2. \quad (1.8)$$

Theorem 1.2 is proved in Section 7.

Marcel Berger's monograph [Be03, pp. 325-353] contains a detailed exposition of the state of systolic affairs up to '03. More recent developments are covered in [Ka07]. Recent publications in systolic geometry include [Be08, Br08a, Br08b, Br08c, DKR08, Ka08, RS08, Sa08, BW09, AK09, KK09, KS09].

## 2. VARIANCE, HERMITE CONSTANT, SUCCESSIVE MINIMA

The proof of inequalities with isosystolic defect relies upon the familiar computational formula for the variance of a random variable in terms of expected values. Keeping our differential geometric application in mind, we will denote the random variable  $f$ . Namely, we have the formula

$$E_\mu(f^2) - (E_\mu(f))^2 = \text{Var}(f), \quad (2.1)$$

where  $\mu$  is a probability measure. Here the variance is

$$\text{Var}(f) = E_\mu((f - m)^2),$$

where  $m = E_\mu(f)$  is the expected value (*i.e.* the mean).

Now consider a flat metric  $\mathcal{G}_0$  of unit area on the 2-torus  $\mathbb{T}^2$ . Denote the associated measure by  $\mu$ . Since  $\mu$  is a probability measure, we can apply formula (2.1) to it. Consider a metric  $\mathcal{G} = f^2\mathcal{G}_0$  conformal to the flat one, with conformal factor  $f(x, y) > 0$ , and new measure  $f^2\mu$ . Then we have

$$E_\mu(f^2) = \int_{\mathbb{T}^2} f^2 \mu = \text{area}(\mathcal{G}).$$

Equation (2.1) therefore becomes

$$\text{area}(\mathcal{G}) - (E_\mu(f))^2 = \text{Var}(f). \quad (2.2)$$

Next, we will relate the expected value  $E_\mu(f)$  to the systole of the metric  $\mathcal{G}$ . To proceed further, we need to deal with some combinatorial preliminaries. We will then relate (2.1) to Loewner's torus inequality.

Let  $B$  be a finite-dimensional Banach space, *i.e.* a vector space together with a norm  $\|\cdot\|$ . Let  $L \subset (B, \|\cdot\|)$  be a lattice of maximal rank, *i.e.* satisfying  $\text{rank}(L) = \dim(B)$ . We define the notion of successive minima of  $L$  as follows.

**Definition 2.1.** For each  $k = 1, 2, \dots, \text{rank}(L)$ , define the  $k$ -th successive minimum of the lattice  $L$  by

$$\lambda_k(L, \|\cdot\|) = \inf \left\{ \lambda \in \mathbb{R} \mid \begin{array}{l} \exists \text{ lin. indep. } v_1, \dots, v_k \in L \\ \text{with } \|v_i\| \leq \lambda \text{ for all } i \end{array} \right\}. \quad (2.3)$$

Thus the first successive minimum,  $\lambda_1(L, \|\cdot\|)$  is the least length of a nonzero vector in  $L$ .

**Definition 2.2.** Let  $b \in \mathbb{N}$ . The *Hermite constant*  $\gamma_b$  is defined in one of the following two equivalent ways:

- (1)  $\gamma_b$  is the *square* of the biggest first successive minimum, *cf.* Definition 2.1, among all lattices of unit covolume;

(2)  $\gamma_b$  is defined by the formula

$$\sqrt{\gamma_b} = \sup \left\{ \frac{\lambda_1(L)}{\text{vol}(\mathbb{R}^b/L)^{1/b}} \mid L \subseteq (\mathbb{R}^b, \|\cdot\|) \right\}, \quad (2.4)$$

where the supremum is extended over all lattices  $L$  in  $\mathbb{R}^b$  with a Euclidean norm  $\|\cdot\|$ .

A lattice realizing the supremum is called a *critical* lattice. A critical lattice may be thought of as the one realizing the densest packing in  $\mathbb{R}^b$  when we place balls of radius  $\frac{1}{2}\lambda_1(L)$  at the points of  $L$ .

### 3. STANDARD FUNDAMENTAL DOMAIN AND EISENSTEIN INTEGERS

**Definition 3.1.** The lattice of the *Eisenstein integers* is the lattice in  $\mathbb{C}$  spanned by the elements 1 and the sixth root of unity.

To visualize the lattice, start with an equilateral triangle in  $\mathbb{C}$  with vertices 0, 1, and  $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ , and construct a tiling of the plane by repeatedly reflecting in all sides. The Eisenstein integers are by definition the set of vertices of the resulting tiling.

The following result is well-known. We reproduce a proof here since it is an essential part of the proof of Loewner's torus inequality with isosystolic defect.

**Lemma 3.2.** *When  $b = 2$ , we have the following value for the Hermite constant:  $\gamma_2 = \frac{2}{\sqrt{3}} = 1.1547\dots$ . The corresponding critical lattice is homothetic to the  $\mathbb{Z}$ -span of the cube roots of unity in  $\mathbb{C}$ , i.e. the Eisenstein integers.*

*Proof.* Consider a lattice  $L \subset \mathbb{C} = \mathbb{R}^2$ . Clearly, multiplying  $L$  by nonzero complex numbers does not change the value of the quotient

$$\frac{\lambda_1(L)^2}{\text{area}(\mathbb{C}/L)}.$$

Choose a “shortest” vector  $z \in L$ , i.e. we have  $|z| = \lambda_1(L)$ . By replacing  $L$  by the lattice  $z^{-1}L$ , we may assume that the complex number  $+1 \in \mathbb{C}$  is a shortest element in the lattice. We will denote the new lattice by the same letter  $L$ , so that now  $\lambda_1(L) = 1$ . Now complete the element  $+1 \in L$  to a  $\mathbb{Z}$ -basis

$$\{\tau, +1\} \quad (3.1)$$

for  $L$ . Thus  $|\tau| \geq \lambda_1(L) = 1$ . Consider the real part  $\Re(\tau)$ . Clearly, we can adjust the basis by adding a suitable integer to  $\tau$ , so as to satisfy

the condition  $-\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}$ . Then the basis vector  $\tau$  lies in the closure of the standard fundamental domain

$$D = \{z \in \mathbb{C} \mid |z| > 1, |\Re(z)| < \frac{1}{2}, \Im(z) > 0\} \quad (3.2)$$

for the action of the group  $\mathrm{PSL}(2, \mathbb{Z})$  in the upperhalf plane of  $\mathbb{C}$ . The imaginary part satisfies  $\Im(\tau) \geq \frac{\sqrt{3}}{2}$ , with equality possible in the following two cases:  $\tau = e^{i\frac{\pi}{3}}$  or  $\tau = e^{i\frac{2\pi}{3}}$ . Finally, we calculate the area of the parallelogram in  $\mathbb{C}$  spanned by  $\tau$  and  $+1$ , and write

$$\frac{\mathrm{area}(\mathbb{C}/L)}{\lambda_1(L)^2} = \Im(\tau) \geq \frac{\sqrt{3}}{2}$$

to conclude the proof.  $\square$

#### 4. FUNDAMENTAL DOMAIN AND LOEWNER'S TORUS INEQUALITY

We now return to the proof of Loewner's torus inequality for the metric  $\mathcal{G} = f^2\mathcal{G}_0$  using the computational formula for the variance. Let us analyze the expected value term  $E_\mu(f) = \int_{\mathbb{T}^2} f\mu$  in (2.2).

By the proof of Lemma 3.2, the lattice of deck transformations of the flat torus  $\mathcal{G}_0$  admits a  $\mathbb{Z}$ -basis similar to  $\{\tau, 1\} \subset \mathbb{C}$ , where  $\tau$  belongs to the standard fundamental domain (3.2). In other words, the lattice is similar to

$$\mathbb{Z}\tau + \mathbb{Z}1 \subset \mathbb{C}.$$

Consider the imaginary part  $\Im(\tau)$  and set

$$\sigma^2 := \Im(\tau) > 0.$$

From the geometry of the fundamental domain it follows that  $\sigma^2 \geq \frac{\sqrt{3}}{2}$ , with equality if and only if  $\tau$  is the primitive cube or sixth root of unity. Since  $\mathcal{G}_0$  is assumed to be of unit area, the basis for its group of deck transformations can therefore be taken to be

$$\{\sigma^{-1}\tau, \sigma^{-1}\},$$

where  $\Im(\sigma^{-1}\tau) = \sigma$ . We will prove the following generalisation of Loewner's bound.

**Theorem 4.1.** *Every metric  $\mathcal{G}$  on the torus satisfies the inequality*

$$\mathrm{area}(\mathcal{G}) - \sigma^2 \mathrm{sys}(\mathcal{G})^2 \geq \mathrm{Var}(f), \quad (4.1)$$

where  $f$  is the conformal factor of the metric  $\mathcal{G}$  with respect to the unit area flat metric  $\mathcal{G}_0$ .

*Proof.* With the normalisations described above, we see that the flat torus is ruled by a pencil of horizontal closed geodesics, denoted  $\gamma_y = \gamma_y(x)$ , each of length  $\sigma^{-1}$ , where the “width” of the pencil equals  $\sigma$ , i.e. the parameter  $y$  ranges through the interval  $[0, \sigma]$ , with  $\gamma_\sigma = \gamma_0$ .

By Fubini’s theorem, we obtain the following lower bound for the expected value:

$$\begin{aligned} E_\mu(f) &= \int_0^\sigma \left( \int_{\gamma_y} f(x) dx \right) dy \\ &= \int_0^\sigma \text{length}(\gamma_y) dy \\ &\geq \sigma \text{sys}(\mathcal{G}), \end{aligned}$$

Substituting into (2.2), we obtain the inequality

$$\text{area}(\mathcal{G}) - \sigma^2 \text{sys}(\mathcal{G})^2 \geq \text{Var}(f), \quad (4.2)$$

where  $f$  is the conformal factor of the metric  $\mathcal{G}$  with respect to the unit area flat metric  $\mathcal{G}_0$ .  $\square$

Since  $\sigma^2 \geq \frac{\sqrt{3}}{2}$ , we obtain in particular a strengthening of Loewner’s torus inequality, namely the following inequality with isosystolic defect:

$$\text{area}(\mathcal{G}) - \frac{\sqrt{3}}{2} \text{sys}(\mathcal{G})^2 \geq \text{Var}(f), \quad (4.3)$$

as discussed in the introduction.

**Corollary 4.2.** *A metric satisfying the boundary case of equality in Loewner’s torus inequality (1.1) is necessarily flat and homothetic to the quotient of  $\mathbb{R}^2$  by the lattice of Eisenstein integers.*

*Proof.* If a metric  $f^2 ds^2$  satisfies the boundary case of equality in (1.1), then the variance of the conformal factor  $f$  must vanish by (4.3). Hence  $f$  is a constant function. The proof is completed by applying Lemma 3.2.  $\square$

Now suppose  $\tau$  is pure imaginary, i.e. the lattice  $L$  is a rectangular lattice of coarea 1. Note that this property for a coarea 1 lattice is equivalent to the equality  $\lambda_1(L)\lambda_2(L) = 1$ .

**Corollary 4.3.** *If  $\tau$  is pure imaginary, then the metric  $\mathcal{G} = f^2 \mathcal{G}_0$  satisfies the inequality*

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f). \quad (4.4)$$

*Proof.* If  $\tau$  is pure imaginary then  $\sigma \geq 1$ , and the inequality follows from (4.2).  $\square$

In particular, every surface of revolution satisfies (4.4), since its lattice is rectangular, cf. Corollary 5.3.



## 5. FIRST FUNDAMENTAL FORM AND SURFACES OF REVOLUTION

This elementary section is concerned mainly with surfaces of revolution and an explicit construction of isothermal coordinates on such surfaces. Recall that the first fundamental form of a regular parametrized surface  $\underline{x}(u^1, u^2)$  in  $\mathbb{R}^3$  is the bilinear form on the tangent plane defined by the restriction of the ambient inner product  $\langle \cdot, \cdot \rangle$ . With respect to the basis  $\{x_1, x_2\}$ , where  $x_i = \frac{\partial \underline{x}}{\partial u^i}$ , it is given by the two by two matrix  $(g_{ij})$ , where  $g_{ij} = \langle x_i, x_j \rangle$  are the metric coefficients.

In the special case of a surface of revolution, it is customary to use the notation  $u^1 = \theta$  and  $u^2 = \varphi$ . The starting point is a curve  $C$  in the  $xz$ -plane, parametrized by a pair of functions  $x = f(\varphi)$ ,  $z = g(\varphi)$ . We will assume that  $f(\varphi) > 0$ . The surface of revolution (around the  $z$ -axis) defined by  $C$  is parametrized as follows:  $\underline{x}(\theta, \varphi) = (f(\varphi) \cos \theta, f(\varphi) \sin \theta, g(\varphi))$ . The condition  $f(\varphi) > 0$  ensures that the resulting surface is an imbedded torus, provided the original curve  $C$  itself is a Jordan curve. The pair of functions  $(f, g)$  gives an arclength parametrisation of the curve if  $\left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2 = 1$ . For example, setting  $f(\varphi) = \sin \varphi$  and  $g(\varphi) = \cos \varphi$ , we obtain a parametrisation of the sphere  $S^2$  in spherical coordinates. To calculate the first fundamental form of a surface of revolution, note that  $x_1 = \frac{\partial \underline{x}}{\partial \theta} = (-f \sin \theta, f \cos \theta, 0)$ , while  $x_2 = \frac{\partial \underline{x}}{\partial \varphi} = \left(\frac{df}{d\varphi} \cos \theta, \frac{df}{d\varphi} \sin \theta, \frac{dg}{d\varphi}\right)$ , so that we have  $g_{11} = f^2 \sin^2 \theta + f^2 \cos^2 \theta = f^2$ , while  $g_{22} = \left(\frac{df}{d\varphi}\right)^2 (\cos^2 \theta + \sin^2 \theta) + \left(\frac{dg}{d\varphi}\right)^2 = \left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2$  and  $g_{12} = -f \frac{df}{d\varphi} \sin \theta \cos \theta + f \frac{df}{d\varphi} \cos \theta \sin \theta = 0$ . Thus we obtain the first fundamental form

$$(g_{ij}) = \begin{pmatrix} f^2 & 0 \\ 0 & \left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2 \end{pmatrix}. \quad (5.1)$$

We have the following obvious lemma.

**Lemma 5.1.** *For a surface of revolution obtained from a unit speed parametrisation  $(f(\varphi), g(\varphi))$  of the generating curve, we obtain the following matrix of the coefficients of the first fundamental form:*

$$(g_{ij}) = \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The following lemma expresses the metric of a surface of revolution in isothermal coordinates.

**Lemma 5.2.** *Suppose  $(f(\varphi), g(\varphi))$ , where  $f(\varphi) > 0$ , is an arclength parametrisation of the generating curve of a surface of revolution. Then*

the change of variable

$$\psi = \int \frac{d\varphi}{f(\varphi)}$$

produces a new parametrisation (in terms of variables  $\theta, \psi$ ), with respect to which the first fundamental form is given by a scalar matrix  $(g_{ij}) = (f^2 \delta_{ij})$ .

In other words, we obtain an explicit conformal equivalence between the metric on the surface of revolution and the standard flat metric on the quotient of the  $(\theta, \psi)$  plane. Such coordinates are referred to as “isothermal coordinates” in the literature. The existence of such a parametrisation is of course predicted by the uniformisation theorem (see Theorem 1.1) in the case of a general surface.

*Proof.* Let  $\varphi = \varphi(\psi)$ . By chain rule,  $\frac{df}{d\psi} = \frac{df}{d\varphi} \frac{d\varphi}{d\psi}$ . Now consider again the first fundamental form (5.1). To impose the condition  $g_{11} = g_{22}$ , we need to solve the equation  $f^2 = \left(\frac{df}{d\psi}\right)^2 + \left(\frac{dg}{d\psi}\right)^2$ , or

$$f^2 = \left( \left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2 \right) \left(\frac{d\varphi}{d\psi}\right)^2.$$

In the case when the generating curve is parametrized by arclength, we are therefore reduced to the equation  $f = \frac{d\varphi}{d\psi}$ , or  $\psi = \int \frac{d\varphi}{f(\varphi)}$ . Replacing  $\varphi$  by  $\psi$ , we obtain a parametrisation of the surface of revolution in coordinates  $(\theta, \psi)$ , such that the matrix of metric coefficients is a scalar matrix.  $\square$

**Corollary 5.3.** *Consider a torus of revolution in  $\mathbb{R}^3$  formed by rotating a Jordan curve with unit speed parametrisation  $(f(\varphi), g(\varphi))$  where  $\varphi \in [0, L]$ , and  $L$  is the total length of the closed curve. Then the torus is conformally equivalent to a flat torus defined by a rectangular lattice*

$$a\mathbb{Z} \oplus b\mathbb{Z},$$

where  $a = 2\pi$  and  $b = \int_0^L \frac{d\varphi}{f(\varphi)}$ .

## 6. A SECOND ISOSYSTOLIC DEFECT TERM

In the notation of Section 3, assume for simplicity that  $\tau = i$ , i.e. the underlying flat metric is that of a unit square torus  $\mathbb{R}^2/\mathbb{Z}^2$  where we think of  $\mathbb{R}^2$  as the  $(x, y)$  plane. For metrics in this conformal class, we will obtain an additional defect term for Loewner’s torus inequality. First, we study a metric  $\mathcal{G} = f^2 ds^2$ , defined by a conformal factor  $f(y) > 0$ , where  $ds^2 = dx^2 + dy^2$  is the standard flat metric and the conformal factor only depends on one of the variables, as in the case

of a surface of revolution, see Section 5. Our estimate is based on the following lemma.

**Lemma 6.1.** *Let  $g$  be a continuous function with zero mean on the unit interval  $[0, 1]$ . Then we have the following bound in terms of the  $L^1$  norm:*

$$\int_0^1 (g - \min_g) \geq \frac{1}{2} |g|_1.$$

*Proof.* Let  $S^+ \subset [0, 1]$  be the set where the function  $g$  is positive, so that  $|g|_1 = \int |g| = 2 \int_{S^+} g$ . Since  $\min_g \leq 0$ , we obtain

$$\int_0^1 (g - \min_g) \geq \int_{S^+} (g - \min_g) \geq \int_{S^+} g = \frac{1}{2} |g|_1,$$

completing the proof of the lemma.  $\square$

Consider the unit square torus  $(\mathbb{R}^2/\mathbb{Z}^2, ds^2)$ , where  $ds^2 = dx^2 + dy^2$ , covered by the  $(x, y)$  plane.

**Theorem 6.2.** *If the conformal factor  $f$  of the metric  $\mathcal{G} = f^2 ds^2$  on  $\mathbb{R}^2/\mathbb{Z}^2$  only depends on one of the two variables, then  $\mathcal{G}$  satisfies the inequality*

$$\text{area}(\mathcal{G}) - \text{Var}(f) \geq (\text{sys}(\mathcal{G}) + \frac{1}{2} |f_0|_1)^2, \quad (6.1)$$

where  $f_0 = f - m$  and  $m$  is the expected value of  $f$ .

To make inequality (6.1) resemble Loewner's torus inequality, we can rewrite it as follows:

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f) + \text{sys}(\mathcal{G}) |f_0|_1 + \frac{1}{4} |f_0|_1^2,$$

so that, in particular, we obtain a form of the inequality which does not involve the systole in the right hand side:

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f) + \frac{1}{4} |f_0|_1^2. \quad (6.2)$$

*Proof of Theorem 6.2.* To fix ideas, assume  $f$  only depends on  $y$ . Let  $y_0$  be the point where the minimum  $\min_f$  of  $f = f(y)$  is attained. The  $\mathcal{G}$ -length of the horizontal unit interval at height  $y_0$  equals

$$\int_0^1 f(x, y_0) dx = \int_0^1 \min_f dx = \min_f. \quad (6.3)$$

Such an interval parametrizes a noncontractible loop on the torus, and we obtain

$$\text{sys}(\mathcal{G}) = \min_f.$$

Applying Lemma 6.1 to  $f_0 = f - E(f)$  where  $f$  is the conformal factor, we obtain

$$E(f) - \text{sys}(\mathcal{G}) = \int_0^1 (f - \min_f) = \int_0^1 (f_0 - \min_{f_0}) \geq \frac{1}{2} |f_0|_1, \quad (6.4)$$

and the theorem follows from (2.2).  $\square$

## 7. BIAXIAL PROJECTION AND SECOND DEFECT

Now consider an arbitrary conformal factor  $f > 0$  on  $\mathbb{R}^2/\mathbb{Z}^2$ . We decompose  $f$  into a sum

$$f(x, y) = E(f) + g_f(x) + h_f(y) + k_f(x, y),$$

where functions  $g_f$  and  $h_f$  have zero means, and  $k_f$  has zero mean along every vertical and horizontal unit interval. The “biaxial” projection  $\mathbb{P}_{\text{BA}}(f)$  is defined by setting

$$\mathbb{P}_{\text{BA}}(f) = g_f(x) + h_f(y). \quad (7.1)$$

In terms of the double Fourier series of  $f$ , the projection  $\mathbb{P}_{\text{BA}}$  amounts to extracting the  $(m, n)$ -terms such that  $mn = 0$  (i.e. the pair of axes), but  $(m, n) \neq (0, 0)$ .

**Theorem 7.1.** *In the conformal class of the unit square torus, the metric  $f^2 ds^2$  defined by a conformal factor  $f(x, y) > 0$ , satisfies the following version of Loewner’s torus inequality with a second defect term:*

$$\text{area}(\mathcal{G}) - \text{sys}(\mathcal{G})^2 \geq \text{Var}(f) + \frac{1}{16} |\mathbb{P}_{\text{BA}}(f)|_1^2. \quad (7.2)$$

*If  $f$  only depends on one variable then the coefficient  $\frac{1}{16}$  in (7.2) can be replaced by  $\frac{1}{4}$ .*

*Proof.* Applying the triangle inequality to (7.1), we obtain

$$|\mathbb{P}_{\text{BA}}(f)|_1 \leq |g_f(x)|_1 + |h_f(y)|_1.$$

Due to the symmetry of the two coordinates, we can assume without loss of generality that

$$|h_f(y)|_1 \geq \frac{1}{2} |\mathbb{P}_{\text{BA}}(f)|_1. \quad (7.3)$$

We define a function  $\bar{f}$  by setting

$$\bar{f}(y) = E(f) + h_f(y) = \int_0^1 f(x, y) dx.$$

We have  $\bar{f} > 0$  since it is an average of a positive function. Clearly, we have  $\bar{f}_0 = h_f$ . By Lemma 6.1 applied to  $\bar{f}_0$ , we obtain

$$\int (\bar{f} - \min \bar{f}) \geq \frac{1}{2} |\bar{f}_0|_1 \geq \frac{1}{4} \mathbb{P}_{\text{BA}}(f)$$

in view of (7.3). We now compare the two metrics  $\bar{f}^2 ds^2$  and  $f^2 ds^2$ . Let  $y_0$  be the point where the function  $\bar{f}$  attains its minimum. Then

$$\text{sys}(\bar{f}^2 ds^2) = \min \bar{f} = \bar{f}(y_0) = \int_0^1 f(x, y_0) dx \geq \text{sys}(f^2 ds^2). \quad (7.4)$$

Meanwhile,

$$E(f) = E(\bar{f}) \geq \text{sys}(\bar{f}^2 ds^2) + \frac{1}{2} |\bar{f}_0|_1 \quad (7.5)$$

by (6.4) applied to the averaged metric  $\bar{f}^2 ds^2$ . Thus,

$$\begin{aligned} \text{area}(f^2 ds^2) - \text{Var}(f) &= E(f)^2 \\ &\geq \left( \text{sys}(\bar{f}^2 ds^2) + \frac{1}{4} \mathbb{P}_{\text{BA}}(f) \right)^2 \\ &\geq \left( \text{sys}(f^2 ds^2) + \frac{1}{4} \mathbb{P}_{\text{BA}}(f) \right)^2 \end{aligned}$$

by combining (7.4) and (7.5).  $\square$

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